and the unit position vector $\hat{r}(t)$ can be written

$$\hat{r}(t)=c_\alpha\hat{A}+s_\alpha\hat{P}(t)=c_\alpha\hat{A}+s_\alpha(c_\delta\hat{R}+s_\delta\hat{S}) \quad (24a)$$

$$\triangleq \rho\hat{r}_0+\sigma\hat{e}_0+\tau\hat{n}_0 \quad (24b)$$

where $\delta=\delta(t)=\Omega t=(\omega_r t)/\cos\alpha$ and

$$\rho=1-s_\alpha^2(1-c_\delta);\quad \sigma=s_\alpha[c_\alpha c_{\psi_0}(1-c_\delta)+s_\delta s_{\psi_0}] \quad (25a,b)$$

$$\tau=-s_\alpha[c_\alpha s_{\psi_0}(1-c_\delta)-s_\delta c_{\psi_0}] \quad (25c)$$

Note that $\hat{r}(t_0)=\hat{r}_0$.

Using the rotation matrices $M_y(-L)$ and $M_z(\lambda)$, which appeared previously in Eq. (4), $\hat{r}(t)$ is transformed to an earth-centered coordinate system

$$\hat{r}(t)=\xi\hat{x}+\eta\hat{y}+\zeta\hat{z} \quad (26)$$

where

$$\zeta=\sin L(t)=\rho s_{L_0}+\tau c_{L_0} \quad (27)$$

and

$$\frac{\eta}{\xi}=\tan\lambda(t)=\frac{\tan\lambda_0+\sigma/(\rho c_{L_0}-\tau s_{L_0})}{1-(\tan\lambda_0)\sigma/(\rho c_{L_0}-\tau s_{L_0})} \quad (28a)$$

$$\triangleq \tan[\lambda_0+\Delta\lambda(t)] \quad (28b)$$

so

$$\tan\Delta\lambda=\sigma/(\rho c_{L_0}-\tau s_{L_0}) \quad (29)$$

Thus, an alternative determination of the ground track is given by Eqs. (27) and (29). As mentioned earlier, this solution is equivalent to the previous solution (which we already know satisfies the differential equations), since both forms of the solution can be transformed one into the other. This transformation is presented in detail in the full paper.

Conclusions

The sole purpose for the preceding development was the need for an analytical description of the ground track for a real aircraft flying in a coordinated turn. In fact, this ground track motion formed the foundation of a target acquisition and pointing program used during a series of flights with an instrumented research aircraft.

On the Wave Drag Integral for Slender Bodies

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Introduction

THE transonic wave drag integral derived from linear theory gives results of engineering accuracy and has therefore found wide application in preliminary design. Of the several methods now available[1-5] for evaluating the drag integral, Eminton's method enjoys some popularity because of its basic simplicity. In this Note we suggest an alternative to the Eminton method and use Filon's quadrature rule to calculate the wave drag, and compare it against the Eminton method and a collocation method.

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Calculation Procedure

The wave drag integral I, for a sufficiently slender body at zero incidence is given by[1]

$$I\equiv D/q=-(\tfrac{1}{2}\pi)\int_0^1\int_0^1 S''(x)S''(\xi)\log|x-\xi|\,dx\,d\xi \quad (1)$$

subject to the auxiliary conditions that

$$S'(0)=S'(1)=0 \quad (2)$$

and that the body be sufficiently smooth so that $S'(x)$ is continuous in $0\leq x\leq 1$. In Eq. (1) D is the wave drag, q the freestream dynamic pressure, $S(x)$ is the body cross-section area distribution, and x is measured from the body nose in the direction of the base along the body axis of symmetry. The body is normalized to unit length and primes denote differentiation with respect to the function argument.

Following Eminton, $S'(x)$ is represented by a Fourier sine series

$$S'(x)=\sum_{r=1}^{\infty}a_r\sin r\theta \quad (3)$$

where

$$x=\tfrac{1}{2}(1-\cos\theta) \quad (4)$$

which automatically satisfies the auxilliary conditions imposed on $S(x)$.

The substitution of Eq. (3) in Eq. (1) results in

$$D/q=(\pi/4)\sum_{r=1}^{\infty}r\,a_r^2 \quad (5)$$

and an integration of Eq. (3) shows

$$S(x)=a_0+\tfrac{1}{4}a_1\theta+\tfrac{1}{4}\sum_{r=2}^{\infty}(a_{r+1}-a_{r-1})\sin r\theta/r \quad (6)$$

The first three coefficients a_0, a_1, and a_2 may be shown to be equal to

$$a_0=S(0) \quad (7)$$

$$a_1=4[S(1)-S(0)]/\pi \quad (8)$$

$$a_2=8[2V-S(1)-S(0)]/\pi \quad (9)$$

where V is the body volume. For a given body the nose area $S(0)$, and the base area $S(1)$ will be known, and frequently V will also be known. The remaining or all of the a_r may be obtained from

$$a_r=(2/\pi)\int_0^\pi S'(x)\sin r\theta\,d\theta;\quad r=1,2,... \quad (10)$$

However, a practical difficulty exists since $S'(x)$ is seldom known in an analytical form or with sufficient accuracy in numerical form at a reasonably large number of points.

To overcome this difficulty, we multiply Eq. (6) by $\sin k\theta$ and integrate over 0 to π to obtain

$$\int_0^\pi S(\theta)\sin k\theta\,d\theta+(\cos k\pi-1)S(0)/k+a_1\pi\cos k\pi/4k$$

$$=\pi(a_{k+1}-a_{k-1})/8k$$

Table 1 Comparison of three methods for calculating the wave drag integral

	Eminton[a]		Collocation		Filton integration	
	10 pts.[b]	15 pts.	7 pts.	15 pts.	51 pts.	91 pts.
a_0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
a_1	12.7324	12.7324	12.7324	12.7324	12.7324	12.7324
a_2	53.0507	53.1129	53.1076	53.1129	53.1270	53.1334
a_3	−14.1194	−14.1880	−14.1910	−14.1880	−14.1902	−14.1943
a_4	39.7020	40.2557	40.1712	40.2557	40.2700	40.2798
a_5	3.4302	3.5228	3.5306	3.5228	3.5280	3.5165
a_6	−21.6988	−22.5804	−23.2030	−22.7008	−22.6954	−22.6754
a_7	0.5372	0.5956	0.6718	0.5990	0.6409	0.5936
a_8	−4.3484	−4.8208		−4.8233	−4.8824	−4.7891
a_9	0.1820	0.2343		0.2364	0.3832	0.2289
a_{10}	−1.7462	−2.0444		−2.0354	−2.2624	−1.9993
a_{11}	0.0883	0.1247		0.1280	0.5534	0.1464
a_{12}	−1.0074	−1.1064		−1.1163	−1.7352	−1.0990
a_{13}	0.0613	0.0862		0.0958	1.0913	0.1671
a_{14}	−0.8781	−0.7004		−0.7730	−2.1060	−0.7656
a_{15}	0.0656	0.0816		0.1161	2.1184	0.2618
$I.10^{-4}$	1.2586[c]	1.2789	1.2802	1.2802	1.2944	1.2802
% Erroy in I	1.67[d]	0.05	−0.05	−0.05	−1.16	−0.05

[a]The first 16 coefficients are tabulated. [b]Refers to the number of points specified for $S(x)$. [c]Exact value of I is 12796. [d]Error in $I = (I \text{ exact} - I) \times 100/I$ exact.

or

$$a_{k+1} = \frac{8k}{\pi}\left\{\int_0^\pi S(\theta)\sin k\theta \,d\theta + S(0)[(-1)^k - 1]/k + a_1\pi(-1)^k/4k\right\} + a_{k-1}; \quad k=1,2,... \tag{11}$$

If the body area distribution is known at a sufficiently large number of equidistant θ-points, the integral in Eq. (11) may be calculated using Filon's quadrature rule,[6] a rule specifically meant for such integrals, and knowing a_0 and a_1 from Eq. (7) and (8), the remaining a_r may be found from Eq. (11). The Filon quadrature rule is given in the Appendix for completeness.

Occasionally, $S(x)$ will be available at unequal θ-intervals in which case any convenient interpolation method (e.g. Lagrangain interpolation) may be used to generate data at equal θ-intervals.

Computational Experience and Discussion

The method developed in the previous section is compared against Eminton's method and also against a collocation solution of Eq. (6) for a body whose area distribution is given by

$$S(x) = 10x^2(400x^4 - 1176x^3 + 1257x^2 - 588x + 108) \tag{12}$$

This body has been used by several authors for numerical comparisons and was originally proposed by Eminton as representative of a high speed fighter. The results for this body are summarized in Table 1, and show that all the three methods are reliable. The Eminton and the collocation methods were found to require double precision arithmetic on an IBM 360/44 whereas it was not necessary for the new method. The new method of this Note has a few advantages: it does not require the simultaneous solution of the a_r and hence computer core requirements are much less compared to the other methods,[1-5] the body area distribution is approximated in a least squares sense by virtue of Eq. (11), and the time requirements are about the same as the Eminton method or the collocation methods for comparable accuracy. In fact each of these methods, on an average, took less than one second on an IBM 360/44 for each column of the results cited in Table 1.

Our computational experience on the new method shows that, in general, if n number of a_r are to be evaluated, about 5n to 6n points on $S(x)$ must be specified. If fewer points are given, additional points must be generated using an interpolation method.

Appendix

For integrals of the form

$$\int_a^b f(t)\cos kt \,dt \quad \text{and} \quad \int_a^b f(t)\sin kt \,dt$$

Filon's quadrature rule states that[6]

$$\int_a^b f(t)\cos kt\,dt \simeq h\{\alpha[f(b)\sin kb - f(a)\sin ka] + \beta C_{2n} + \gamma C_{2n-1}\}$$

$$\int_a^b f(t)\sin kt\,dt \simeq h\{-\alpha[f(b)\cos kb - f(a)\cos ka] + \beta S_{2n} + \gamma S_{2n-1}\}$$

where

$$C_{2n} = \tfrac{1}{2}f(a)\cos ka + f(a+2h)\cos k(a+2h) + f(a+4h)\cos k(a+4h) + ... + \tfrac{1}{2}f(b)\cos kb$$

$$C_{2n-1} = f(a+h)\cos k(a+h) + f(a+3h)\cos k(a+3h) + ... + f(b-h)\cos k(b-h)$$

$$S_{2n} = \tfrac{1}{2}f(a)\sin ka + f(a+2h)\sin k(a+2h) + f(a+4h)\sin k(a+4h) + ... + \tfrac{1}{2}f(b)\sin kb$$

$$S_{2n-1} = f(a+h)\sin k(a+h) + f(a+3h)\sin k(a+3h) + ... + f(b-h)\sin k(b-h)$$

$$\alpha = (\theta^2 + \theta\sin\theta\cos\theta - 2\sin^2\theta)/\theta^3$$

$$\beta = 2[\theta(1+\cos^2\theta) - 2\sin\theta\cos\theta]/\theta^3$$

$\gamma \quad = 4(\sin\theta - \theta \cos\theta)/\theta_3$

$\theta \quad = kh = k(b-a)2N$

$h \quad = (b-a)2N$, where the interval (a,b) is divided into $2N$ subintervals of equal length.

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Experimental Study of Axial Flow in Wing Tip Vortices

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Introduction

THE vortex generated behind each wing tip of a fixed-wing aircraft may be hazardous to following aircraft,[1] whereas the vortex generated behind the tip of a helicopter rotor blade may interact with a following blade causing noise[2] and vibration.[3] A knowledge of the structure of a tip vortex is a necessary prerequesite to attempts to alleviate these problems.

The importance of axial flow in a tip vortex and of its interaction with the tangential flowfield has been investigated theoretically.[4,5] Experimental measurements in the wind tunnel[6-8] and in flight[9-11] have shown widely differing axial velocity distributions within the vortex core, with velocity excesses being found in some cases and deficits in others. This note describes a qualitative towing tank study of some of the factors which control the axial flowfield in a trailing vortex and attempts an explanation of the differing results of previous axial velocity measurements.

Experimental Details

The experiments were conducted in a small towing tank ($0.3 \times 0.3 \times 4$ m). Two rectangular wing models were used with NACA 6412 and NACA 0012 sections. Each wing had a span of 125 mm and a chord of 64 mm, and could be mounted, with its spanwise axis vertical, below the towing carriage. Most tests were carried out at Reynolds numbers of 3.4×10^4 and 6.8×10^4, based on wing chord.

Three alternative tip configurations were tested for each wing. The basic tip was cut square, normal to the spanwise axis, and the rounded tip had a semicircular cross section at each chordwise station. A sharp edged tip was formed by extending the basic wing spanwise, using a thin brass plate cambered to the upper surface contour. The spanwis dimension of this extension plate was equal to the maximum wing thickness.

The hydrogen bubble technique was used for flow visualization. A horizontal cathode wire spanning the tank normal to the direction of wing motion generated a vertical curtain of bubbles between the wire and the water surface. As the wing passed through the curtain, axial flow was indicated by bubbles leaving the plane of the curtain. Bubbles moving in the same direction as the wing indicated a region of velocity deficit in a frame of reference fixed to the wing (as in a wind tunnel, for example), while bubbles moving in the opposite direction indicated a velocity excess in the wing fixed reference frame.

An alternative cathode consisted of a chordwise strip of aluminum foil 2 mm wide fixed to the wing upper surface at the tip. Bubbles generated by this cathode marked the core of the vortex behind the wing.

Results and Discussion

For all the wings tested, at angles of incidence of less than approximately 10°, the axial-flow pattern showed a velocity deficit on the core centerline at all stations behind the wing. At the core edges, a small velocity excess appeared at about 10-15 chord lengths behind the wing. The core edge excess persisted until approximately 30 chord lengths behind the wing before merging into an axial velocity deficit across the whole core.

At angles of incidence of 10-20°, the axial-flow patterns varied for the different wings. In the case of the square tipped NACA 6412 section wing an axial velocity excess appeared on the core centerline immediately behind the wing at an angle of incidence of approximately 10°. The magnitude of the centerline excess increased with increasing incidence. At some point downstream of the wing, the centerline velocity excess changed suddenly to a deficit. The changeover point was identifiable as the point at which bubbles from the bubble curtain which had been moving in the opposite direction to the wing suddenly reversed their motion (Fig. 1). The changeover point was also visible, using the wing tip cathode, as a region of sudden expansion of the vortex core, similar in appearance to a vortex burst. The changeover point moved closer to the wing with increasing incidence and decreased Reynolds number, as shown in Fig. 2.

The configuration of the wing tip affected the position of the axial-flow changeover point, as also shown in Fig. 2. Compared to the square tipped wing, rounding the tip edge shifted the point closer to the wing. For the sharp edged wing tip, the changeover occurred further from the wing at most angles of incidence.

Behind the NACA 0012 section wing, no velocity excess on the core centerline was detected at any incidence or for any tip configuration. Thus, the aerofoil section may be significant in determining the axial-flow distribution in the vortex core. Whether this effect is due to details of the tip flow for the different sections or to variations in the overall wing characteristics is not clear.

In the incidence range 10-20°, the subsequent flow development for all wing configurations was similar to that for angles of incidence below 10°, i.e., a small core edge excess appeared and later merged into a deficit across the whole core.

More than 30-35 chord lengths behind all the wings tested, at all angles of incidence, regions of axial velocity excess appeared at random along the core and moved erratically upstream and downstream. Each of these regions terminated in a sudden reversion to an axial velocity deficit, again similar in

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